

# Quark condensate and deviations from string-like behaviour of meson spectra

S.S. Afonin

V.A. Fock Department of Theoretical Physics, St. Petersburg State University, Russia  
e-mail: afonin24@mail.ru, Sergey.Afonin@pobox.spbu.ru

## Abstract

I analyse the hypothesis that deviations from the linear meson mass spectra appear due to the dynamical chiral symmetry breaking in QCD. It is shown that the linear mass spectrum for the light, non-strange vector and axial-vector mesons is then parametrized by the constant  $f_\pi$ , being successful phenomenologically. The toy model for deviations from linearity is proposed.

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## 1 Introduction

It is widely believed that Quantum Chromodynamics and the boson string theory are tightly connected. In the latter, the meson mass spectrum  $M^2(n)$  is linear with respect to the number of radial excitation  $n$ . Phenomenological analysis of the experimental data shows that this picture, at least qualitatively, indeed takes place [1]. Nevertheless, in the real world there are sizeable deviations from linear trajectories. There arises the natural question: what is the dynamics behind these deviations from the string model? Partly, the deviations can be induced by mixing of resonances and thresholds effects. Let us exclude these effects by setting the widths of resonances to zero. In the QCD it is equivalent to the limit  $N_c = \infty$  [2]. This limit is a good approximation to the real world, having an accuracy of about 10%. However, even in that limit one expects the linear mass spectrum only at a large enough  $n$ , as it is in the two-dimensional multicolor QCD — the 't Hooft model [3].

In the present work I propose and analyze the hypothesis that, at least in the large- $N_c$  limit, deviations from the linear mass spectrum arise due to the dynamical chiral symmetry breaking in QCD at low energies.

The convenient tools for the theoretical analysis are the two-point correlation functions for the quark currents. In Euclidean space there exist the Operator Product Expansion (OPE) for them at a large euclidean momentum [4]. Their OPE results in the set of so-called chiral symmetry restoration (CSR) sum rules. Making use of these sum rules for the case of light, non-strange vector (V) and axial-vector (A) mesons, I first analyze the consequences of the hypothesis for the linear mass spectrum (sect. 2) and then consider a toy model describing the corrections to the linear trajectories (sect. 3). I conclude in sect. 4.

## 2 Linear mass spectrum

Let us consider the vector two-point functions which are defined in Euclidean space as

$$\Pi_{\mu\nu}^J = \int d^4e^{iQx} \langle \bar{q}\Gamma_\mu^J q(x) \bar{q}\Gamma_\nu^J q(0) \rangle = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi^J(Q^2). \quad (1)$$

There  $\Gamma_\mu^V = \gamma_\mu$ ,  $\Gamma_\mu^A = \gamma_\mu \gamma_5$ , and throughout the paper  $J = V, A$ . In the chiral limit the axial-vector correlator does not have the longitudinal part due to the pion pole.

In the chiral and large- $N_c$  limits, the OPE for the objects under consideration is [4]

$$\Pi^J(Q^2) = \frac{1}{4\pi^2} \left( 1 + \frac{\alpha_s}{\pi} \right) \ln \frac{\Lambda^2}{Q^2} + \frac{\alpha_s}{12\pi} \frac{\langle G^2 \rangle}{Q^4} + \frac{4a^J}{9} \pi \alpha_s \frac{\langle \bar{q}q \rangle^2}{Q^6} + \mathcal{O} \left( \frac{1}{Q^8} \right), \quad (2)$$

where  $a^V = -7$ ,  $a^A = 11$ ,  $\Lambda$  is a normalization scale,  $\langle G^2 \rangle$  and  $\langle \bar{q}q \rangle$  are the gluon and quark condensate respectively. For simplicity, the first-order perturbative correction to the partonic logarithm will not be taken into account. As follows from Eq. (2), the difference  $\Pi^V - \Pi^A$  (equal to zero in the perturbation theory) is proportional to  $\langle \bar{q}q \rangle^2 / Q^6$ . This reflects the fact that the quark condensate is an order parameter of the chiral symmetry breaking in QCD and that the chiral symmetry is restored at high energies.

On the other hand, the functions  $\Pi^J(Q^2)$  satisfy the dispersion relation

$$\Pi^J(Q^2) = \int_0^\infty \frac{ds}{s + Q^2} \frac{1}{\pi} \text{Im} \Pi^J(s) + \text{Subt. const.} \quad (3)$$

If one assumes the string-like picture and the color confinement in the large- $N_c$  QCD, the meson spectrum consists then of the infinite set of equidistant narrow resonances with an universal slope in all channels. These states saturate completely the imaginary part of the correlation function:

$$\frac{1}{\pi} \text{Im} \Pi^J(s) = 2f_\pi^2 \delta(s) \delta_{A,J} + 2 \sum_{n=0}^\infty F_J^2(n) \delta(s - M_J^2(n)). \quad (4)$$

In Eq. (4)  $f_\pi$  is the pion decay constant,  $f_\pi \approx 90$  MeV (in the chiral limit),  $M_J(n)$  and  $F_J$  are the corresponding masses and decay constants,  $\delta_{A,J} = 1$  for  $J = A$  and is zero otherwise. The first term in the l.h.s. of Eq. (4) is due to the contribution of the pion pole to the axial-vector correlator. If all quantum numbers are fixed the meson spectrum has the linear parametrization in the string-like picture:

$$M_J^2(n) = m_J^2 + \mu^2 n, \quad F_J^2 = \text{const}, \quad n = 0, 1, \dots, \quad (5)$$

where  $m_J^2$  is an intercept,  $\mu^2$  is an universal slope, and  $n$  is the number of radial excitation. According to the made hypothesis and the discussion above, corrections to the spectrum (5) are proportional to the quark condensate. We omit them in the analysis of this section. Thus, the two-point functions take the form

$$\Pi^J(Q^2) = \frac{2f_\pi^2}{Q^2} \delta_{A,J} + \sum_{n=0}^\infty \frac{2F_J^2}{Q^2 + m_J^2 + \mu^2 n} = \frac{2f_\pi^2}{Q^2} \delta_{A,J} - \frac{2F_J^2}{\mu^2} \psi \left( \frac{Q^2 + m_J^2}{\mu^2} \right), \quad (6)$$

where the irrelevant (infinite) constants are cancelled. The  $\psi$  function has an asymptotic representation at  $z \gg 1$

$$\psi(z) = \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}, \quad (7)$$

there  $B_{2n}$  denote the Bernoulli numbers.

Comparing expansion in  $1/Q^2$  (7) for Eq. (6) with the OPE (2), one arrives at the following CSR sum rules:

$$\frac{1}{8\pi^2} = \frac{F_J^2}{\mu^2}, \quad (8)$$

$$f_\pi^2 \delta_{A,J} = F_J^2 \left( \frac{m_J^2}{\mu^2} - \frac{1}{2} \right), \quad (9)$$

$$\frac{\alpha_s}{12\pi} \langle G^2 \rangle = F_J^2 \mu^2 \left( \frac{m_J^4}{\mu^4} - \frac{m_J^2}{\mu^2} + \frac{1}{6} \right), \quad (10)$$

$$0 = -\frac{2}{3} F_J^2 \mu^4 \left( \frac{m_J^6}{\mu^6} - \frac{3}{2} \frac{m_J^4}{\mu^4} + \frac{1}{2} \frac{m_J^2}{\mu^2} \right). \quad (11)$$

The l.h.s. of Eq. (11) is set to zero by virtue of the made hypothesis: the parameters of linear mass spectrum (5) do not depend on the quark condensate, which appears in the l.h.s. of Eq. (11) only after allowance for the corrections to the spectrum (5). This factorization is crucial for the given analysis.

Let us set  $m_J^2 = x\mu^2$ . Eq. (11) has the solutions:  $x_1 = 0$ ,  $x_2 = 1/2$ , and  $x_3 = 1$ . As follows from Eq. (9), the solution  $x_2$  corresponds to the vector channel and for the axial-vector case one has the only possibility — the solution  $x_3$ . The solution  $x_1$  may be relevant to the pseudoscalar channel which we do not consider here.

From Eqs. (8) and (9) one has for the decay constants and slope<sup>1</sup>

$$F_J = \sqrt{2} f_\pi, \quad \mu = 4\pi f_\pi. \quad (12)$$

Thus, within this model, the whole V,A meson spectrum is parametrized by only one constant  $f_\pi$ . Following the classification of solutions given above, one obtains ( $n = 0, 1, \dots$ ):

$$\begin{aligned} M_V^2(n) &= 16\pi^2 f_\pi^2 \left( \frac{1}{2} + n \right), \\ M_A^2(n) &= 16\pi^2 f_\pi^2 (1 + n). \end{aligned} \quad (13)$$

The two remarkable properties of the spectrum (13) should be indicated: the value of the  $\rho$  meson mass,  $M_\rho = 2\sqrt{2}\pi f_\pi$ , which is highly successful prediction with respect to experiment [5], and the natural generalization of the Weinberg relation [6] ( $M_{a_1} = \sqrt{2}M_\rho$ )

$$M_A^2(n) = M_V^2(n) + M_\rho^2. \quad (14)$$

The comparison of model estimates and experimental data is presented in Table 1. The data agree well within the large- $N_c$  approximation.

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<sup>1</sup>Here and further I do not trace the exact  $N_c$ -dependence of the physical quantities. In particular, the factor  $\sqrt{3/N_c}$  for the slope  $\mu$  is omitted in (12).

The ensuing from Eq. (10) estimate for the value of gluon condensate exceeds the corresponding phenomenological value,  $\frac{\alpha_s}{12\pi}\langle G^2 \rangle = (360 \text{ MeV})^4$ , in the axial-vector channel,  $\frac{\alpha_s}{12\pi}\langle G^2 \rangle = 64\pi^2 f_\pi^4 = (450 \text{ MeV})^4$ , and gives a negative value in the vector one. The difficulty with the gluon condensate in the sum rule (10) is known (see the first paper in [8]). Within the scheme under consideration this problem can be overcome by introducing some non-linear corrections to the spectrum (5).

### 3 Non-linear mass spectrum

Let us consider the following ansatz for the meson mass spectrum:

$$M_J^2(n) = m_J^2 + \mu^2 n + \frac{f_\pi^2 d_J}{(n+1)^{k_J}}, \quad k_J > 2. \quad (15)$$

By hypothesis, the last term in Eq. (15) (where the dimensional parameter is singled out for clarity) represents a decreasing in  $n$  correction to the linear spectrum, caused by the chiral symmetry breaking. Labeling this contribution as  $\delta$ , one may write (the index  $J$  is omitted):

$$\sum_n \frac{F^2}{Q^2 + m^2 + \mu^2 n + \delta} = \sum_n \frac{F^2}{Q^2 + m^2 + \mu^2 n} - \sum_n \frac{F^2 \delta}{(Q^2 + m^2 + \mu^2 n)(Q^2 + m^2 + \mu^2 n + \delta)}. \quad (16)$$

The first term in the r.h.s. of Eq. (16) can be treated as in the previous section. In the second one, the summation over  $n$  and the expansion in  $1/Q^2$  can be permuted by virtue of the absolute convergence (up to  $\mathcal{O}(1/Q^4)$ ).

Let us consider the axial-vector channel. The sum rules (8), (9), and (11) are solved in the zero order of the condensate expansion. Due to Eqs. (2) and (15), in the first order of this expansion Eq. (11) takes the form:

$$d_A = \frac{11\pi\alpha_s}{8\pi^2\zeta(k_A - 1)} \frac{\langle \bar{q}q \rangle^2}{f_\pi^6} + \mathcal{O}\left(\frac{\langle \bar{q}q \rangle^4}{f_\pi^{12}}\right), \quad (17)$$

where  $\zeta(x)$  is the Riemann zeta function. Let us impose the condition of fulfillment of the sum rule (10) in the first order of the condensate expansion. This yields the equation for the parameter  $k_A$ :

$$\frac{1}{f_\pi^4} \frac{\alpha_s}{\pi} \langle G^2 \rangle = 64\pi^2 - 48\zeta(k_A) d_A. \quad (18)$$

In the vector channel the corresponding equations look as follows:

$$d_V = -\frac{7\pi\alpha_s}{2\pi^2 [2\zeta(k_V - 1) - \zeta(k_V)]} \frac{\langle \bar{q}q \rangle^2}{f_\pi^6} + \mathcal{O}\left(\frac{\langle \bar{q}q \rangle^4}{f_\pi^{12}}\right), \quad (19)$$

$$\frac{1}{f_\pi^4} \frac{\alpha_s}{\pi} \langle G^2 \rangle = -32\pi^2 - 48\zeta(k_V) d_V.$$

For the inputs  $f_\pi = 90 \text{ MeV}$ ,  $\langle \bar{q}q \rangle = -(240 \text{ MeV})^3$ ,  $\frac{\alpha_s}{\pi} \langle G^2 \rangle = (360 \text{ MeV})^4$ , and  $\alpha_s = 0.3$  the numerical calculations give  $k_A \approx k_V \approx 2.1$  and show that the quark condensate terms

indeed can be treated as small parameters (their numerical contributions are less than the large- $N_c$  counting). The presented solution lowers the mass of  $\rho$  meson by 40 MeV and enhances that of  $a_1$  meson by 20 MeV. The masses of excitations virtually do not change.

For the given fits, the correction to the Weinberg relation is

$$M_{a_1}^2 - 2M_\rho^2 \approx \frac{\langle \bar{q}q \rangle^2}{20f_\pi^4}, \quad (20)$$

that is, in this toy model the deviation from the Weinberg relation represents an order parameter of the chiral symmetry breaking in QCD.

## 4 Summary

In the present work I have considered the problem of matching the vector and axial-vector two-point correlation functions in the large- $N_c$  and chiral limits to their Operator Product Expansion at a large euclidean momentum  $Q^2$ . In the first limit the two-point functions are saturated by an infinite number of narrow resonances and, given an ansatz for the meson spectrum, can be calculated exactly. Expanding a correlator in powers of  $Q^{-2}$  and comparing with its OPE, one obtains the sum rule at each order of the expansion. This activity was dealt with by various authors in the vector channel [9], in the vector and axial-vector channels on the same footing [8], and in the vector, axial-vector, scalar, and pseudoscalar channels simultaneously [10].

The distinguishing feature of the present analysis consists in the underlying hypothesis that corrections to the string-like meson spectrum are related to the chiral symmetry breaking in QCD. These contributions are proportional to the quark condensate squared (in the first order of the condensate expansion) and can be treated as small parameters. The sum rule (s.r.) at  $Q^{-6}$ , which is sensitive to the chiral symmetry breaking, gives then the set of solutions for the intercepts (in the zero order of the condensate expansion). The s.r. at  $Q^{-2}$  allows to classify these solutions. The s.r. at  $Q^0$  yields the residues (decay constants) and the s.r. at  $Q^{-2}$  in the axial-vector channel gives the value of universal slope. As a result, the linear meson mass spectrum turns out to be parametrized by only one constant  $f_\pi$ , being successful phenomenologically. The s.r. at  $Q^{-4}$  serves for checking the solutions. It is found that this s.r. can not be satisfied without non-linear corrections to the string-like spectrum. These corrections turned out to be rather small, but they make the sum rules self-consistent.

Thus, a simple toy-model is proposed, where the string-like part of QCD spectrum and the part responsible for the chiral symmetry breaking are naturally factorized. The phenomenological success of the model may signify that the considered physical picture is related to the real QCD.

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$n$	1	2	3	4
$M_V(n)$	800 (769.8 $\pm$ 0.8)	1390 (1465 $\pm$ 25)	1790 (1700 $\pm$ 20)	2120 (2149 $\pm$ 17)
$M_A(n)$	1130 (1230 $\pm$ 40)	1600 (1640 $\pm$ 40)	1960	2260
$\mu = 1130 (1090 \div 1140) [1], \quad F_{a_1} = 130 (123 \pm 25), \quad F_\rho = 130 (154 \pm 8)$				

Table 1: The linear meson mass spectrum (in MeV) for  $f_\pi = 90$  MeV. The known experimental values [7] are displayed in brackets.